Models for Probabilistic Programs with an Adversary

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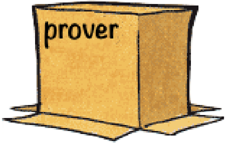
University of Pennsylvania

Probabilistic Programming Semantics 2016

Interactive Proofs



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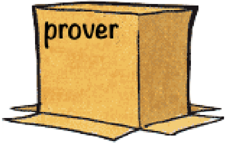

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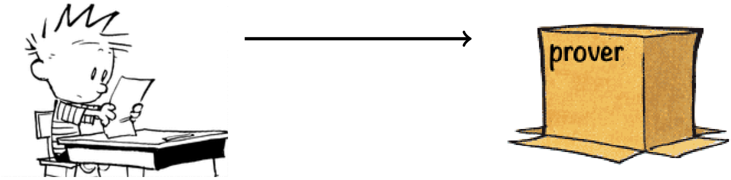

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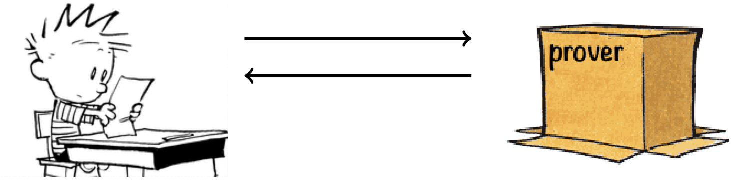

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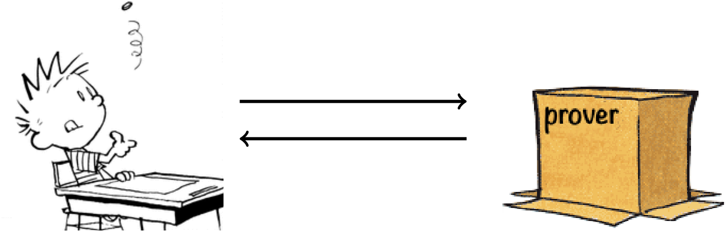

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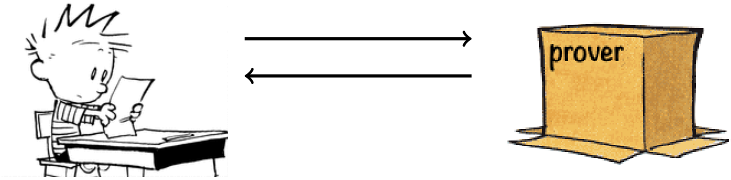

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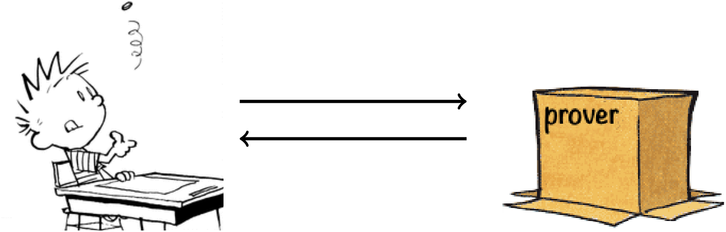

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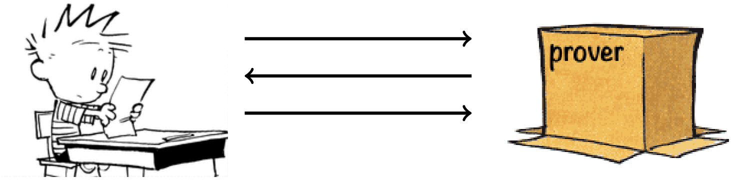

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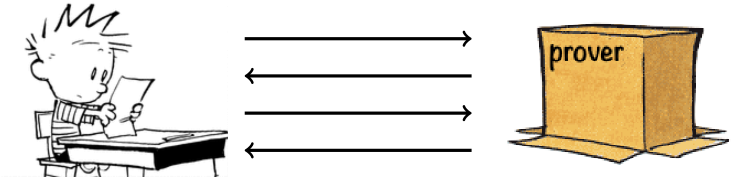
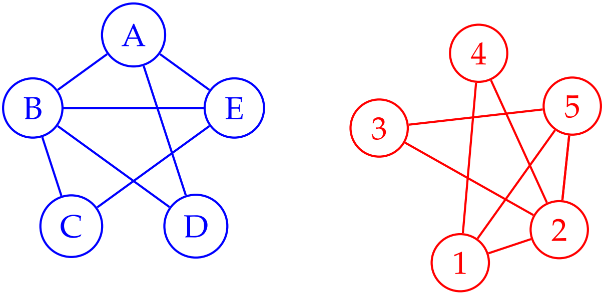


Graph Non-Isomorphism



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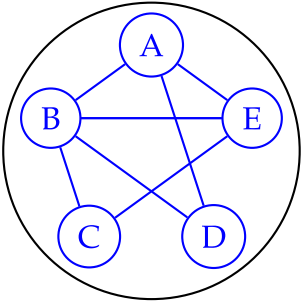

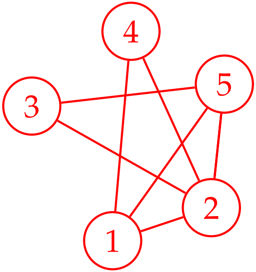

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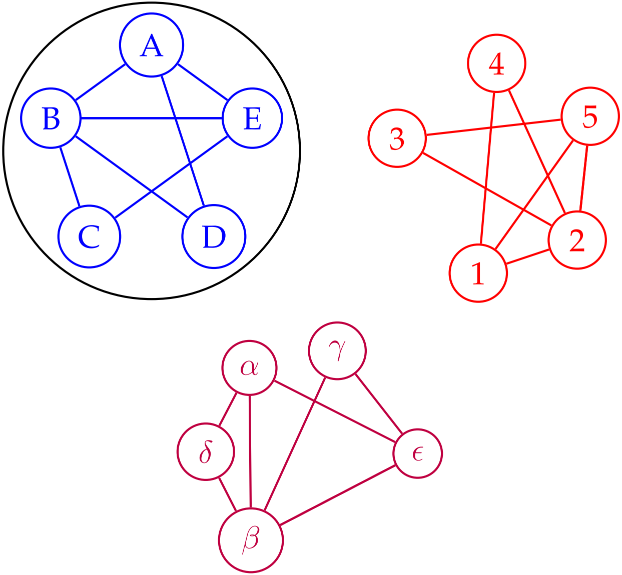

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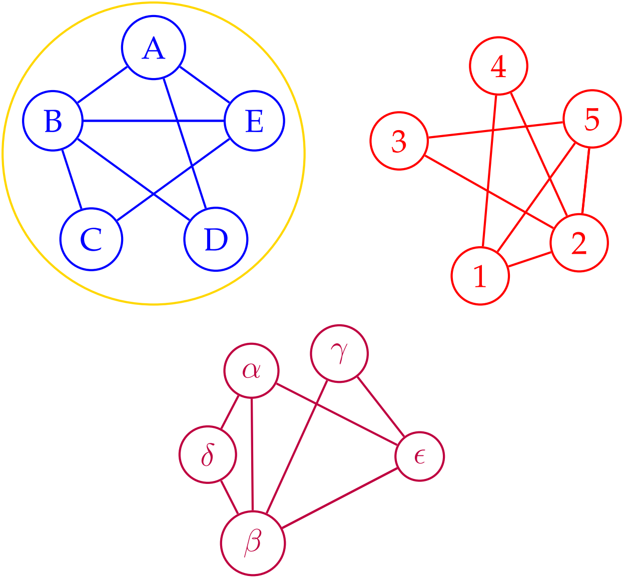
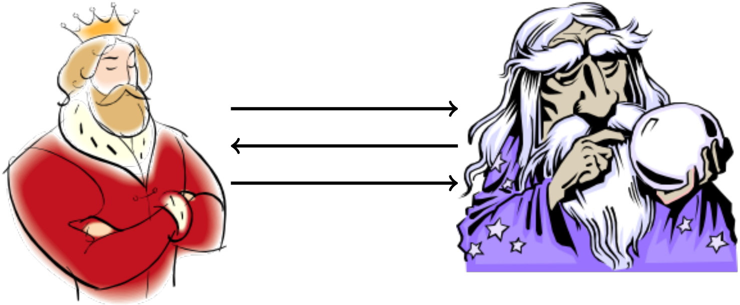


Arthur Merlin Games



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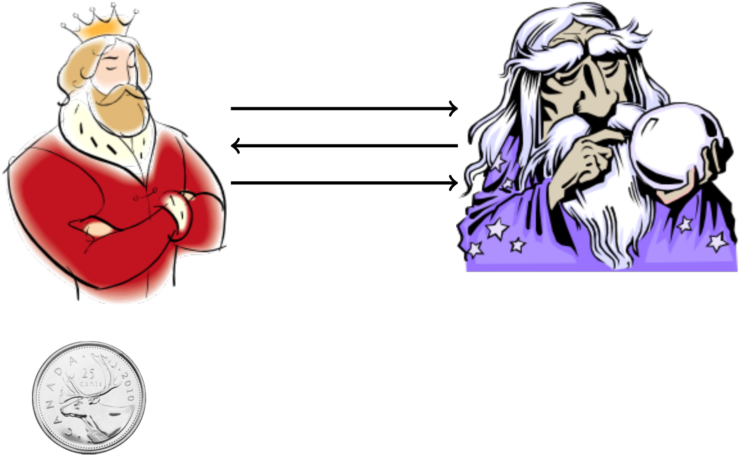

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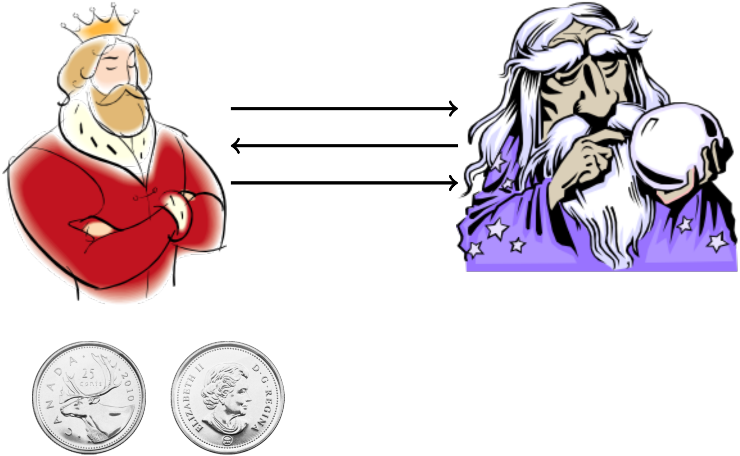

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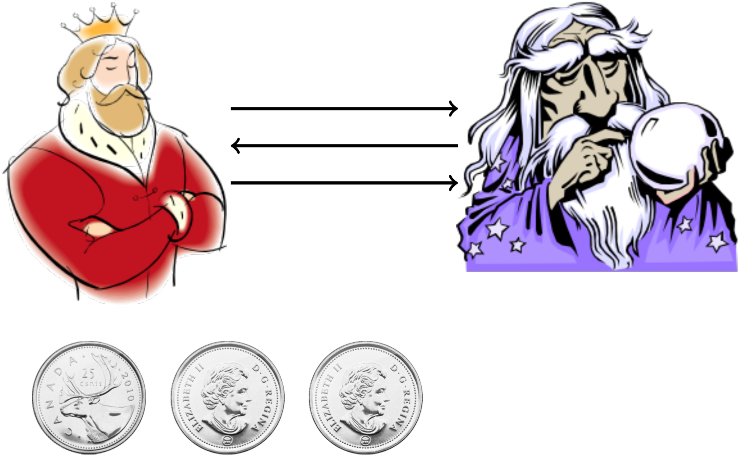

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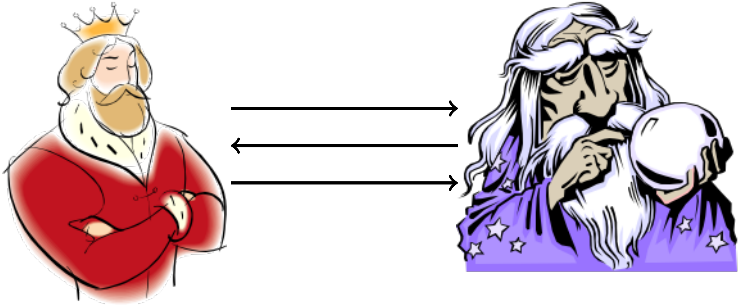

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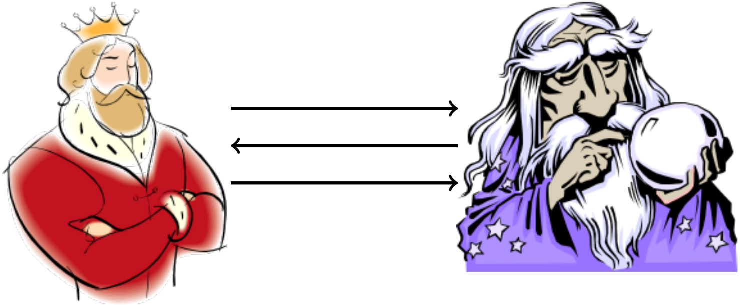


Why Should We Care?

- Mixing probability and nondeterminism is powerful.
- Private vs. public coins matter.

Let's Start with a Deterministic Semantics...

$$\frac{}{\texttt{skip} \ / \ \sigma \Downarrow \sigma} \qquad \frac{\sigma(a) = n}{x \ \texttt{:=} \ a \ / \ \sigma \Downarrow \sigma[x \mapsto n]}$$

$$\frac{c_1 \ / \ \sigma \Downarrow \sigma' \qquad c_2 \ / \ \sigma' \Downarrow \sigma''}{c_1; c_2 \ / \ \sigma \Downarrow \sigma''}$$

$$\frac{\sigma(b) = \texttt{T} \qquad c_1 \ / \ \sigma \Downarrow \sigma'}{\texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2 \ / \ \sigma \Downarrow \sigma'}$$

For Point Distributions

$$\boxed{\Theta ::= [\sigma] \mid \Theta \oplus_p \Theta}$$

$$\frac{}{\texttt{skip} \;/\; [\sigma] \Downarrow [\sigma]} \qquad \frac{[\sigma](a) = n}{x \texttt{ := } a \;/\; [\sigma] \Downarrow [\sigma[x \mapsto n]]}$$

$$\frac{c_1 \;/\; [\sigma] \Downarrow \Theta \qquad c_2 \;/\; \Theta \Downarrow \Theta'}{c_1; c_2 \;/\; [\sigma] \Downarrow \Theta'}$$

$$\frac{\sigma(b) = \texttt{T} \qquad c_1 \;/\; [\sigma] \Downarrow \Theta}{\texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2 \;/\; [\sigma] \Downarrow \Theta}$$

Toss in Some Probability

$$\boxed{\Theta ::= [\sigma] \mid \Theta \oplus_p \Theta}$$

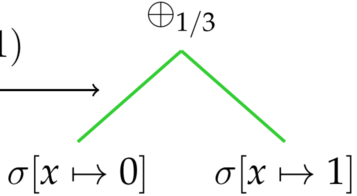
$$\frac{c_1 \ / \ [\sigma] \Downarrow \Theta_1 \qquad c_2 \ / \ [\sigma] \Downarrow \Theta_2}{(c_1 \ \oplus_p \ c_2) \ / \ [\sigma] \Downarrow \Theta_1 \oplus_p \Theta_2}$$

Toss in Some Probability

$$\boxed{\Theta ::= [\sigma] \mid \Theta \oplus_p \Theta}$$

$$\frac{c_1 \;/\; [\sigma] \Downarrow \Theta_1 \qquad c_2 \;/\; [\sigma] \Downarrow \Theta_2}{(c_1 \;\oplus_p\; c_2) \;/\; [\sigma] \Downarrow \Theta_1 \oplus_p \Theta_2}$$

$$[\sigma] \xrightarrow{\quad (x \;:=\; 0 \;\oplus_{\frac{1}{3}}\; x \;:=\; 1) \quad}$$

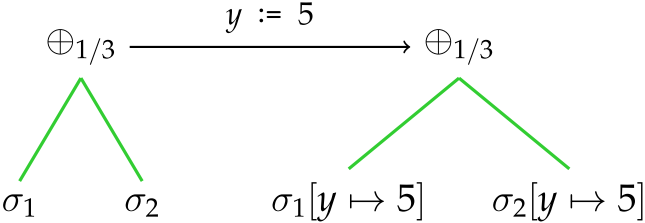


And Lift!

$$\frac{c \;/\; \Theta_1 \Downarrow \Theta_1' \qquad c \;/\; \Theta_2 \Downarrow \Theta_2'}{c \;/\; \Theta_1 \oplus_p \Theta_2 \Downarrow \Theta_1' \oplus_p \Theta_2'}$$

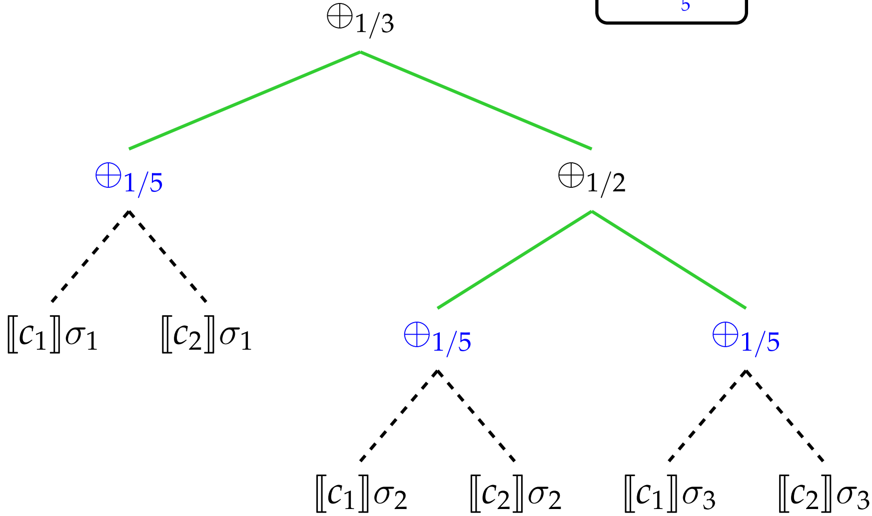
And Lift!

$$\frac{c \;/\; \Theta_1 \Downarrow \Theta_1' \qquad c \;/\; \Theta_2 \Downarrow \Theta_2'}{c \;/\; \Theta_1 \oplus_p \Theta_2 \Downarrow \Theta_1' \oplus_p \Theta_2'}$$

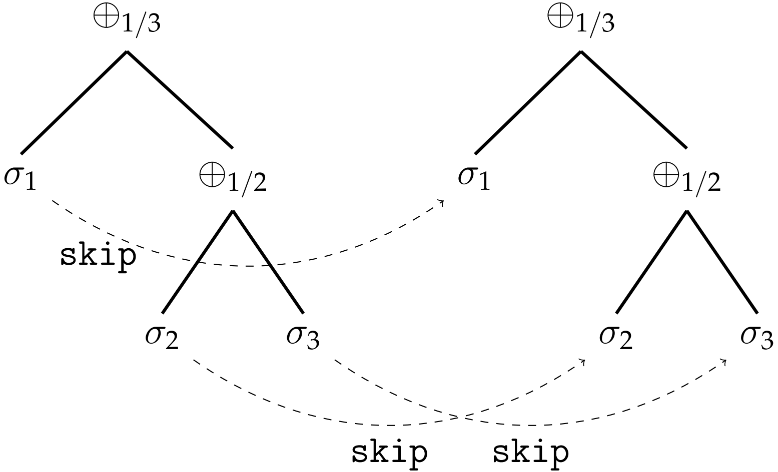


The Toss Command

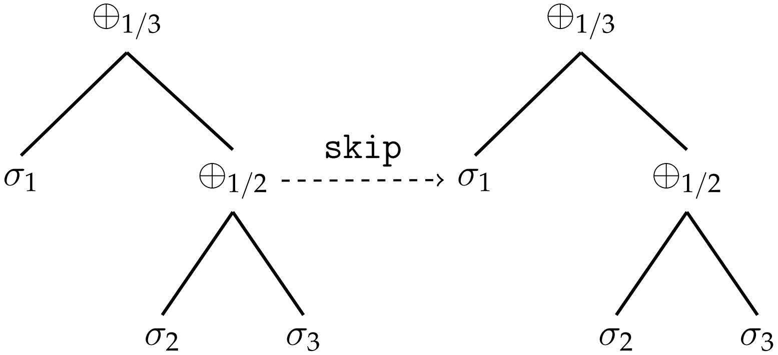
$$\boxed{c_1 \oplus_{\frac{1}{5}} c_2}$$



The Skip Command



More Direct



$\oplus_{1/3}$
σ_1
$\oplus_{1/2}$
σ_2
σ_3
skip
$\oplus_{1/3}$
σ_1
$\oplus_{1/2}$
σ_2
σ_3

Direct Semantics

$$\frac{}{\texttt{skip} \;/\; \Theta \Downarrow \Theta} \qquad \frac{\sigma(a) = n}{x \;\texttt{:=}\; a \;/\; \Theta \Downarrow \Theta[\sigma_i(x) \mapsto n]}$$

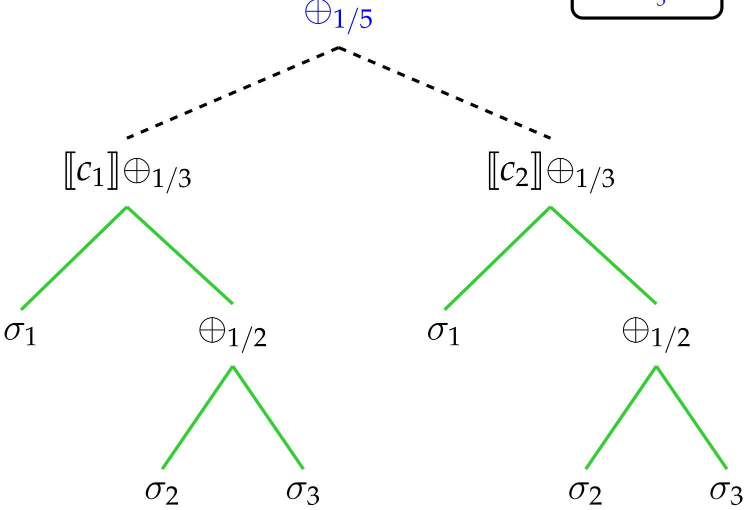
$$\frac{c_1 \;/\; \Theta \Downarrow \Theta' \qquad c_2 \;/\; \Theta' \Downarrow \Theta''}{c_1; c_2 \;/\; \Theta \Downarrow \Theta''}$$

$$\frac{Pr_b(\Theta_1) = 1 \quad c_1 \;/\; \Theta_1 \Downarrow \Theta_1' \qquad c_2 \;/\; \Theta_0 \Downarrow \Theta_0' \quad Pr_b(\Theta_0) = 0}{\texttt{if}\; b \;\texttt{then}\; c_1 \;\texttt{else}\; c_2 \;/\; \Theta_1 \oplus_p \Theta_0 \Downarrow \Theta_1' \oplus_p \Theta_0'}$$

$$\frac{c_1 \;/\; \Theta \Downarrow \Theta_1 \qquad c_2 \;/\; \Theta \Downarrow \Theta_2}{(c_1 \;\oplus_p\; c_2) \;/\; \Theta \Downarrow \Theta_1 \oplus_p \Theta_2}$$

Direct Toss

$\boxed{c_1 \oplus_{\frac{1}{5}} c_2}$



The Distinction

Recursive

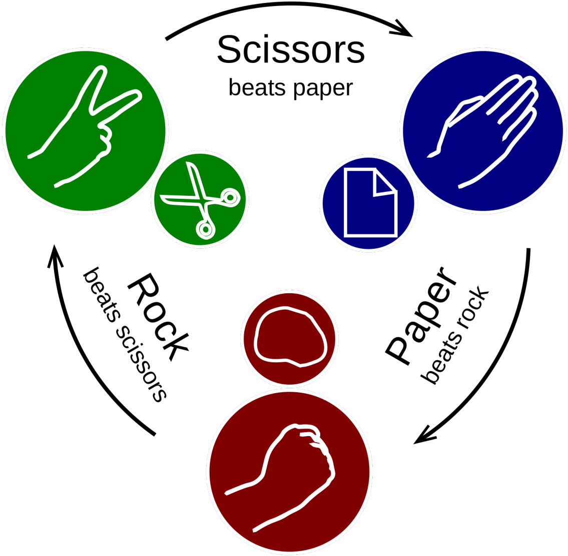
$$\frac{c_1 \;/\; [\sigma] \Downarrow \Theta_1}{(c_1 \;\sqcup\; c_2) \;/\; [\sigma] \Downarrow \Theta_1} \qquad \frac{c_2 \;/\; [\sigma] \Downarrow \Theta_2}{(c_1 \;\sqcup\; c_2) \;/\; [\sigma] \Downarrow \Theta_2}$$

vs.

$$\frac{c_1 \;/\; \Theta \Downarrow \Theta_1}{(c_1 \;\sqcup\; c_2) \;/\; \Theta \Downarrow \Theta_1} \qquad \frac{c_2 \;/\; \Theta \Downarrow \Theta_2}{(c_1 \;\sqcup\; c_2) \;/\; \Theta \Downarrow \Theta_2}$$

Direct

Let's Play a Game!



Let's Play a Game!

$P :=$  $\oplus_{\frac{1}{3}}$ $\big($  $\oplus_{\frac{1}{2}}$  $\big)$

$O :=$  $\sqcup$  $\sqcup$ 

Let's Play a Game!

c_1 $\quad P :=$  $\oplus_{\frac{1}{3}}$ ( $\oplus_{\frac{1}{2}}$ )

c_2 $\quad O :=$  $\sqcup$  $\sqcup$ 

Direct Play

c_1 : $P :=$  $\oplus_{\frac{1}{3}}$ ( $\oplus_{\frac{1}{2}}$ )

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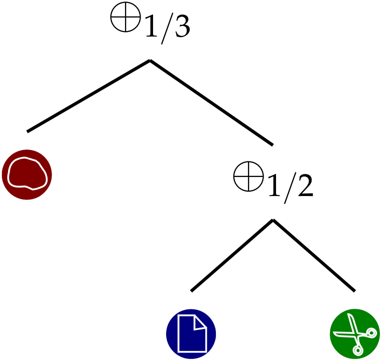
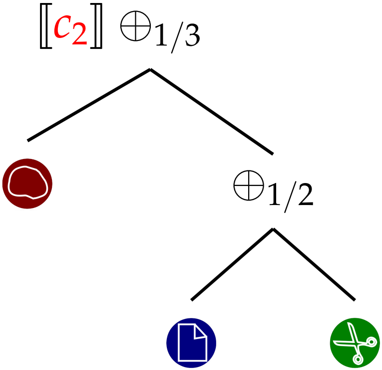


Direct Play

$c_2 : O :=$ 



Direct Play

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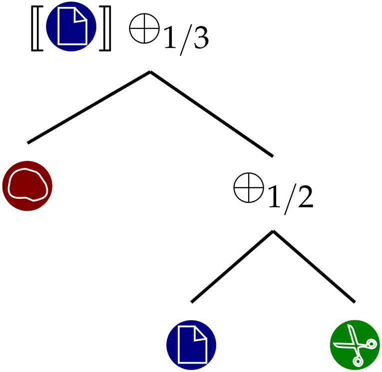
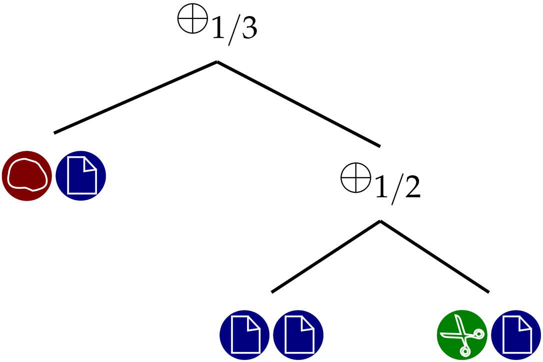


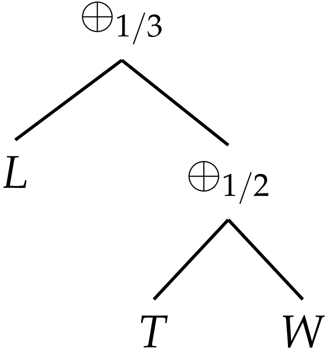
Direct Play

$c_2 : O :=$ 
$\sqcup$
$\sqcup$



Direct Play

$c_2 : O :=$  $\sqcup$  $\sqcup$ 



Recursive Play

c_1 : P:=  $\oplus_{\frac{1}{3}}$ ( $\oplus_{\frac{1}{2}}$ )

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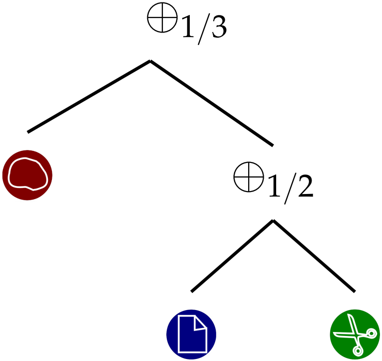
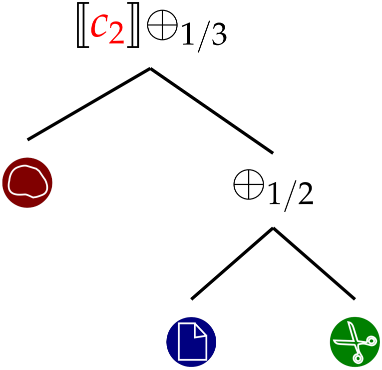


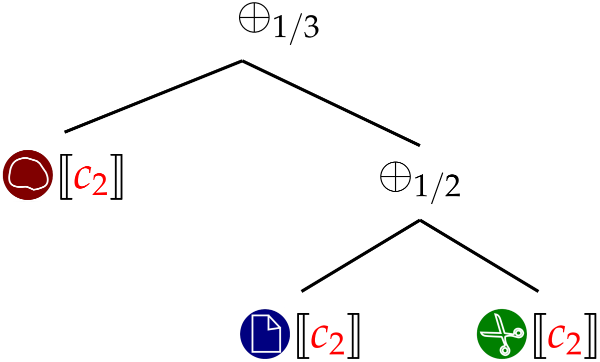
Recursive Play

$c_2 : O :=$ 



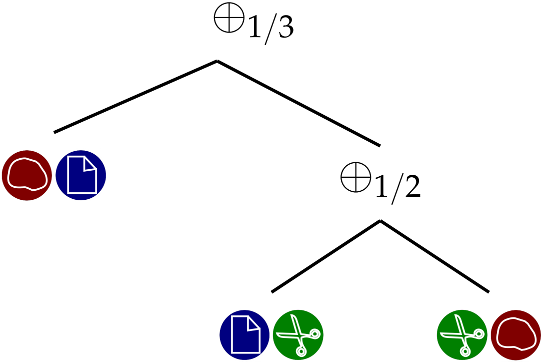
Recursive Play

$c_2 : O :=$ ⊔ ⊔



Recursive Play

$c_2 : O :=$ 



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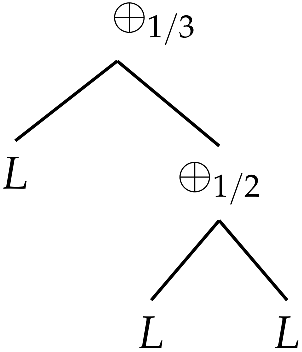

Knowledge

The two levels of operational semantics reflect whether the adversary knows the outcome of coin flips.

Levels of Knowledge

1. Adversary is blind to probabilistic outcomes.
   - Single choice in $((c_1 \sqcup c_2) \oplus (c_1 \sqcup c_2))$
   - Distinct choices in $((c_1 \sqcup c_2) \oplus (c_1 \sqcup c_2))$ (Direct)
2. Adversary can see current program state
3. Adversary recalls program history (Recursive)
4. Adversary can foresee all outcomes.
   - Single coin flip in $((c_1 \oplus c_2) \sqcup (c_1 \oplus c_2))$
   - Distinct coin flips in $((c_1 \oplus c_2) \sqcup (c_1 \oplus c_2))$

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So...

What can we verify?

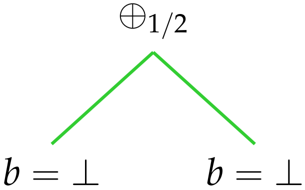
Verification: Direct

$$\frac{\{P\}\ c_1\ \{Q\} \qquad \{P\}\ c_2\ \{Q\}}{\{P\}\ (c_1 \sqcup c_2)\ \{Q\}}$$

Verification: Recursive

$$\frac{\{True\}\ b\ \mathtt{:=}\ \mathtt{T}\ \{Pr(b) = 1\} \qquad \{True\}\ b\ \mathtt{:=}\ \mathtt{F}\ \{Pr(b) = 0\}}{\{True\}\ (b\ \mathtt{:=}\ \mathtt{T} \sqcup b\ \mathtt{:=}\ \mathtt{F})\ \{Pr(b) = 1 \vee Pr(b) = 0\}}$$

Verification: Recursive



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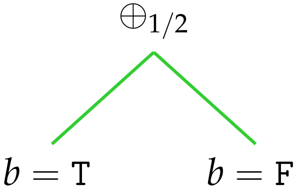
Verification: Recursive



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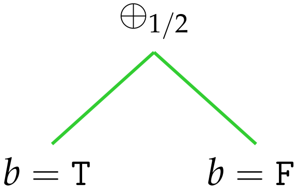

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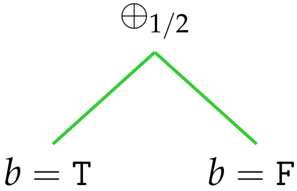
Verification: Recursive



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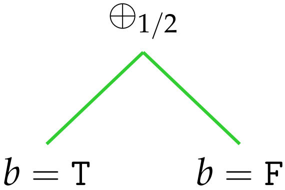
Q cannot include disjunctions

Verification: Recursive

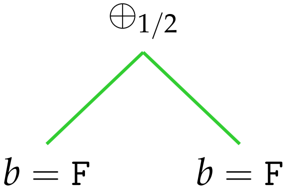
$$\frac{\{Pr(b) = \frac{1}{2}\}\ \texttt{skip}\ \{Pr(b) = \frac{1}{2}\} \qquad \{Pr(b) = \frac{1}{2}\}\ b\ \texttt{:=}\ \neg b\ \{Pr(b) = \frac{1}{2}\}}{\{Pr(b) = \frac{1}{2}\}\ (\texttt{skip} \sqcup b\ \texttt{:=}\ \neg b)\ \{Pr(b) = \frac{1}{2}\}}$$

Verification: Recursive



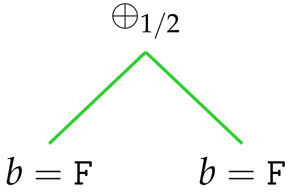
$$\frac{\{Pr(b) = \tfrac{1}{2}\}\ \mathtt{skip}\ \{Pr(b) = \tfrac{1}{2}\} \qquad \{Pr(b) = \tfrac{1}{2}\}\ b\ \mathtt{:=}\ \neg b\ \{Pr(b) = \tfrac{1}{2}\}}{\{Pr(b) = \tfrac{1}{2}\}\ (\mathtt{skip}\ \sqcup\ b\ \mathtt{:=}\ \neg b)\ \{Pr(b) = \tfrac{1}{2}\}}$$

Verification: Recursive



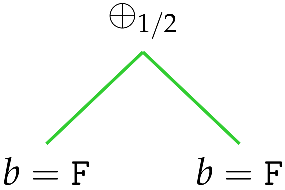
$$\frac{\{Pr(b) = \frac{1}{2}\}\ \mathtt{skip}\ \{Pr(b) = \frac{1}{2}\} \qquad \{Pr(b) = \frac{1}{2}\}\ b\ \mathtt{:=}\ \neg b\ \{Pr(b) = \frac{1}{2}\}}{\{Pr(b) = \frac{1}{2}\}\ (\mathtt{skip} \sqcup b\ \mathtt{:=}\ \neg b)\ \{Pr(b) = \frac{1}{2}\}}$$

Verification: Recursive



$$\frac{\{Pr(b) = \tfrac{1}{2}\}\ \mathtt{skip}\ \{Pr(b) = \tfrac{1}{2}\} \qquad \{Pr(b) = \tfrac{1}{2}\}\ b \mathrel{\mathtt{:=}} \neg b\ \{Pr(b) = \tfrac{1}{2}\}}{\{Pr(b) = \tfrac{1}{2}\}\ (\mathtt{skip} \sqcup b \mathrel{\mathtt{:=}} \neg b)\ \{Pr(b) = \tfrac{1}{2}\}}$$

Verification: Recursive



$$\frac{\{Pr(b) = \frac{1}{2}\}\ \mathtt{skip}\ \{Pr(b) = \frac{1}{2}\} \qquad \{Pr(b) = \frac{1}{2}\}\ b\ \mathtt{:=}\ \neg b\ \{Pr(b) = \frac{1}{2}\}}{\{{\color{red}Pr(b) = \frac{1}{2}}\}\ (\mathtt{skip} \sqcup b\ \mathtt{:=}\ \neg b)\ \{Pr(b) = \frac{1}{2}\}}$$

P cannot include probabilities in $(0, 1)$

Verification: Recursive

$$\frac{\{P\}\ c_1\ \{Q\} \qquad \begin{array}{c}\text{non-probabilistic } P \\ \text{non-disjunctive } Q\end{array} \qquad \{P\}\ c_2\ \{Q\}}{\{P\}\ (c_1 \sqcup c_2)\ \{Q\}}$$

Compositionality

$$(c_1 \;\sqcup\; c_2); (c_3 \;\sqcup\; c_4)$$

Compositionality

$$\{P\}\ (c_1 \ \sqcup \ c_2); (c_3 \ \sqcup \ c_4)\ \{R\}$$

Compositionality

$$\{P\}\ (c_1 \ \sqcup\ c_2)\ \{Q\}\ (c_3 \ \sqcup\ c_4)\ \{R\}$$

Compositionality

$$\{P\}\ (c_1 \ \sqcup \ c_2)\ \{Q\}\ (c_3 \ \sqcup \ c_4)\ \{R\}$$

Compositionality

non-probabilistic P
non-disjunctive Q

$$\{P\}\ (c_1 \sqcup c_2)\ \{Q\}\ (c_3 \sqcup c_4)\ \{R\}$$

Compositionality

non-probabilistic P
non-disjunctive Q

$$\{P\}\ (c_1\ \sqcup\ c_2)\ \{Q\}\ (c_3\ \sqcup\ c_4)\ \{R\}$$

Compositionality

non-probabilistic P non-probabilistic Q
non-disjunctive Q non-disjunctive R

$$\{P\}\ (c_1 \ \sqcup \ c_2)\ \{Q\}\ (c_3 \ \sqcup \ c_4)\ \{R\}$$

Compositionality

non-probabilistic P non-probabilistic Q

non-disjunctive Q non-disjunctive R

$$\{P\}\ (c_1 \sqcup c_2)\ \{Q\}\ (c_3 \sqcup c_4)\ \{R\}$$

Are private coins applicable?

Theorem (Minimax Theorem)

For every two-person, zero-sum game with finitely many strategies, there exists a value V *and a mixed strategy for each player, such that*

1. Given player 2's strategy, the best payoff possible for player 1 is V*, and*

2. Given player 1's strategy, the best payoff possible for player 2 is $-V$*.*

Game Theory

- game $\Longleftrightarrow$ program with nondeterminism

Game Theory

- game $\iff$ program with nondeterminism
- zero sum $\iff$ returns a single value

Game Theory

- game $\iff$ program with nondeterminism
- zero sum $\iff$ returns a single value
- finitely many strategies $\iff$ no unbounded loops

Game Theory

- game $\iff$ program with nondeterminism
- zero sum $\iff$ returns a single value
- finitely many strategies $\iff$ no unbounded loops
- mixed strategy $\iff$ choice of p, q, r annotating the $\oplus$s

Game Theory

Theorem (Minimax Theorem Restated)

Any finite program combining probability and nondeterminism with a single output value has a dual program with the probabilistic and nondeterministic choices inverted, that returns the same value in the worst case.

Game Theory Questions

- Can we use this to find and prove Nash Equilibria in games?
- Does this yield useful generalizations of Nash Equilibrium?
- Can we discover useful compositionality results from this formulation?

More Open Questions

- How does a semantics using infinite bit streams compare to our distribution semantics?
- Can we enumerate the possible interactions between probability and nondeterminism via algebraic equivalences?
  - Can we extend KAT to probabilistic-nondeterministic programs?
- Can we translate between Direct and Recursive Semantics?

Questions?

Thank You

Questions?
Answers?

Questions?
Answers?
Rebuttals?